CDI – II – Prática F7 19/4/21

Ficha 7:

Coordenadas Polares em $\mathbb{R}^2$:

(r,θ)

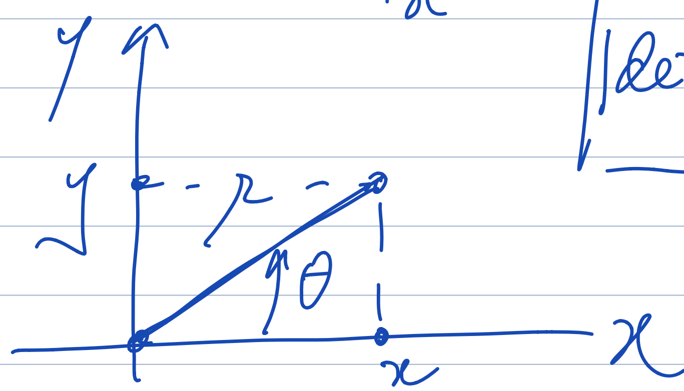
$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$

$$\Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan \dfrac{y}{x} \end{cases}$$

$(r,\theta) \xrightarrow{g} (x,y)$

$$\boxed{|\det Dg(r,\theta)| = r}$$

$$\boxed{dx\,dy \;\to\; r\,dr\,d\theta}$$



$$\int_X f(x)\,dx = \int_T f(g(t))\,|\det Dg(t)|\,dt$$

$$g : T \subset \mathbb{R}^n \longrightarrow X \subset \mathbb{R}^n$$

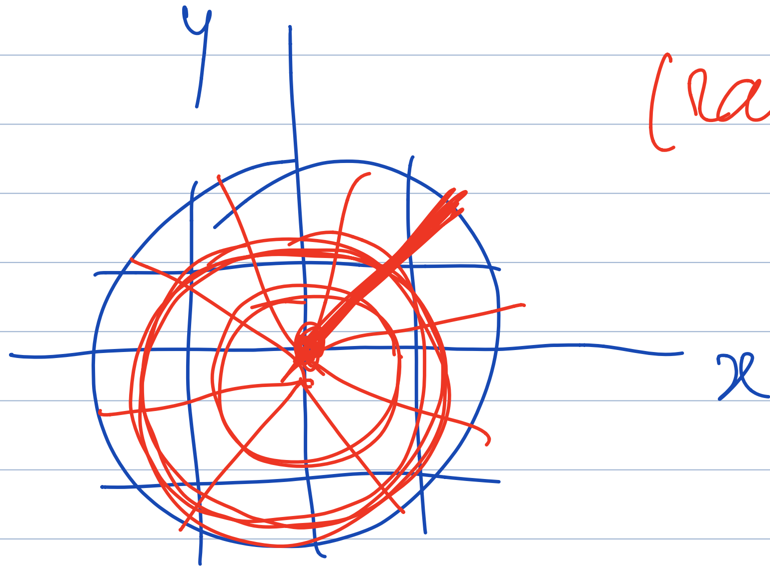
$$t \longmapsto g(t) = x$$

(substitui x por t)

$\underline{\mathbb{R}^2}$: $(r, \theta) \longmapsto (x, y)$

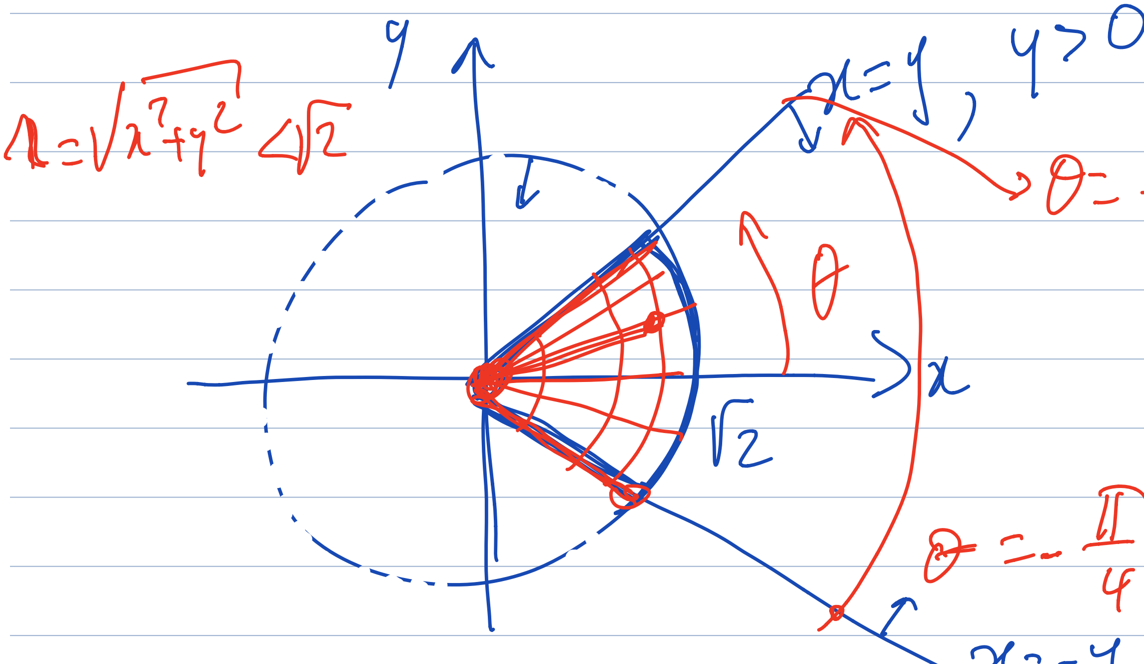
$$g(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$T \left|\begin{array}{l} r > 0 \\ 0 < \theta < 2\pi \end{array}\right.$$



(Raios, arcos de circunf.).

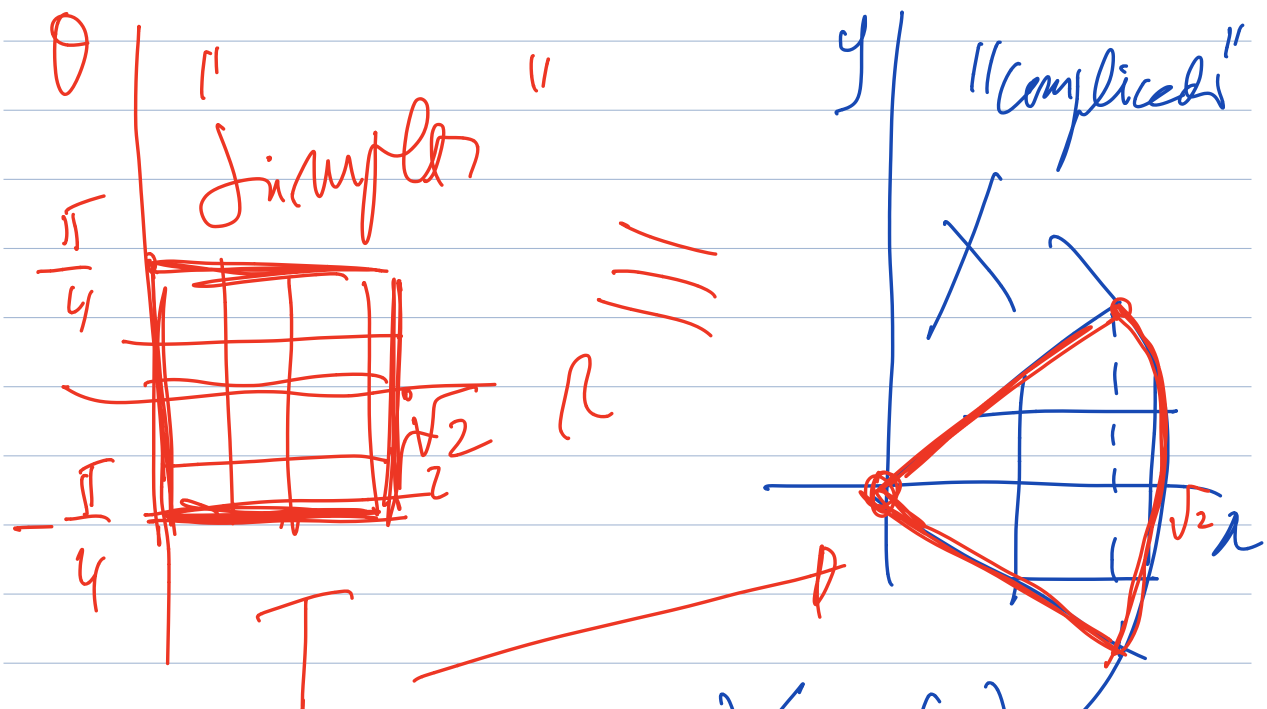
1-a) $x^2+y^2<2 \; , \; x>|y|$



$y<0$

$0<r<\sqrt{2}$

$-\frac{\pi}{4}<\theta<\frac{\pi}{4}$



$(x,y) = g(r,\theta)$ $\qquad$ $X = g(T)$

$\approx$ $dx\,dy$

$$\iint_X f(x,y)\,dx\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\int_0^{\sqrt{2}} f(g(r,\theta))\, r\,dr\right)d\theta$$

1-c) $\quad 0 \le x \le 1 \;;\; -\sqrt{1-x^2} \le y \le x$

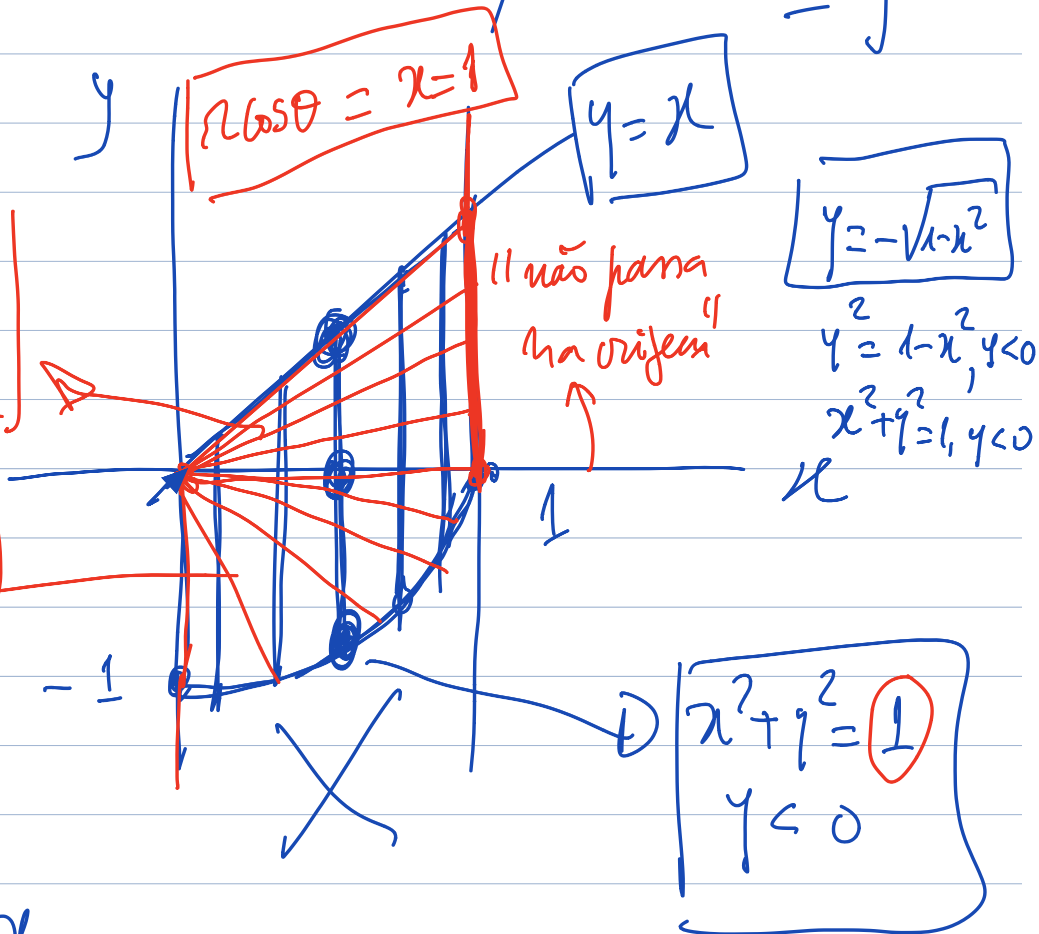
$dy\,dx$

$0 < \theta < \frac{\pi}{4}$

$0 < r < \frac{1}{\cos\theta}$

$0 < r < 1$

$-\frac{\pi}{2} < \theta < 0$



$$\int_0^1 \left(\int_{-\sqrt{1-x^2}}^{x} f(x,y)\,dy \right) dx$$

1 integral dupla.

$$\int_{-\frac{\pi}{2}}^{0} \left(\int_0^1 f(g(r,\theta))\, r\,dr \right) d\theta + \int_0^{\pi/4} \left(\int_0^{\frac{1}{\cos\theta}} f(g(r,\theta))\, r\,dr \right) d\theta$$

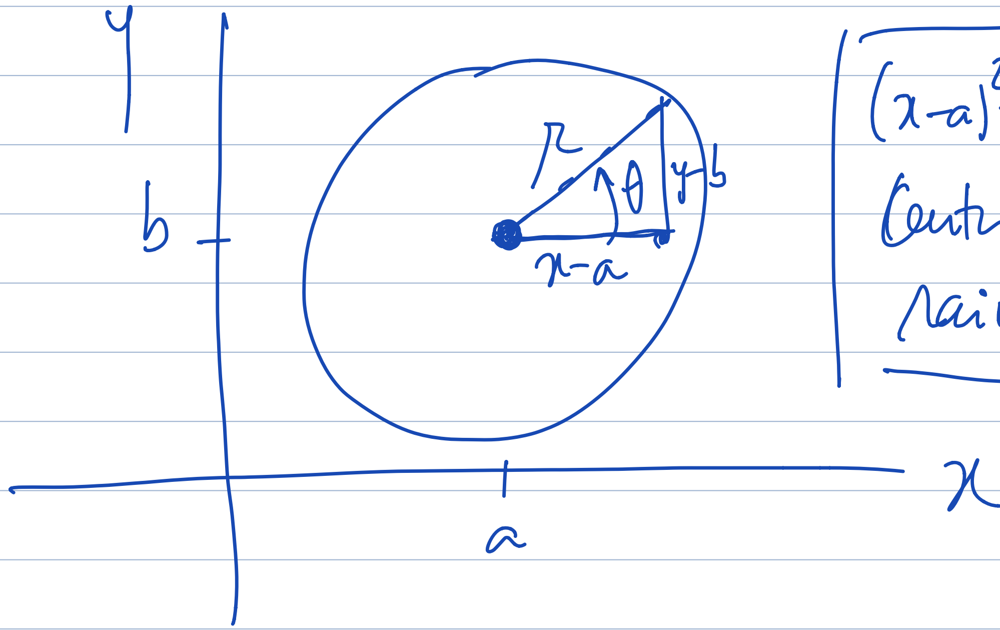
Nota: Para calcular, por exemplo, a área de X devemos usar (r,θ) para $y<0$ e (x,y) para $y>0$.

$$\int_{-\frac{\pi}{2}}^{0}\left(\int_{0}^{1} r\,dr\right)d\theta + \int_{0}^{1}\left(\int_{0}^{x} dy\right)dx$$

$$\underline{\qquad\qquad} \parallel \underline{\qquad\qquad}$$

2 – $\boxed{\text{Coordenadas polares modificadas}}$

1 — Centro do círculo $\neq (0,0)$



$$\boxed{\begin{array}{l}(x-a)^2+(y-b)^2<R^2\\ \text{Centro } (a,b)\\ \text{raio } R\end{array}}$$

$$(x-a)^2+(y-b)^2<R^2$$

$$\begin{cases} x-a = r\cos\theta \\ y-b = r\,\mathrm{sen}\,\theta \end{cases} \Longrightarrow \boxed{\begin{array}{l} 0<r<R \\ 0<\theta<2\pi \end{array}}$$

$$\begin{cases} x = a + R\cos\theta \\ y = b + r\,\mathrm{sen}\,\theta \end{cases}$$

$$g(r,\theta) = (x, y)$$

$$= (a + r\cos\theta,\ b + r\,\mathrm{sen}\,\theta)$$

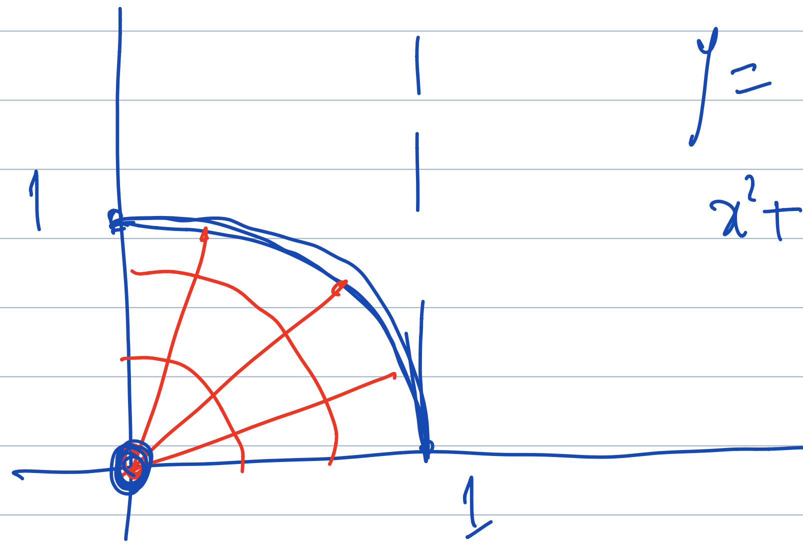
$$\boxed{\det Dg(r,\theta) = r}$$

2-a) $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1-x^2} \end{cases}$

$$f(x,y) = e^{-x^2-y^2}$$

$$\int_0^1 \left(e^{-x^2} \int_0^{\sqrt{1-x^2}} e^{-y^2}\, dy \right) dx$$

primitiva ✗ ⟶ Mudar (r,θ)



$y = \sqrt{1-x^2} \geq 0$

$x^2 + y^2 = 1, \; y \geq 0$

$$\begin{cases} 0 < r < 1 \\ 0 < \theta < \frac{\pi}{2} \end{cases}$$

r

$$-\frac{1}{2}\int_0^{\pi/2}\left(\int_0^1 (-2)r\,e^{-r^2}\,dr\right)d\theta =$$

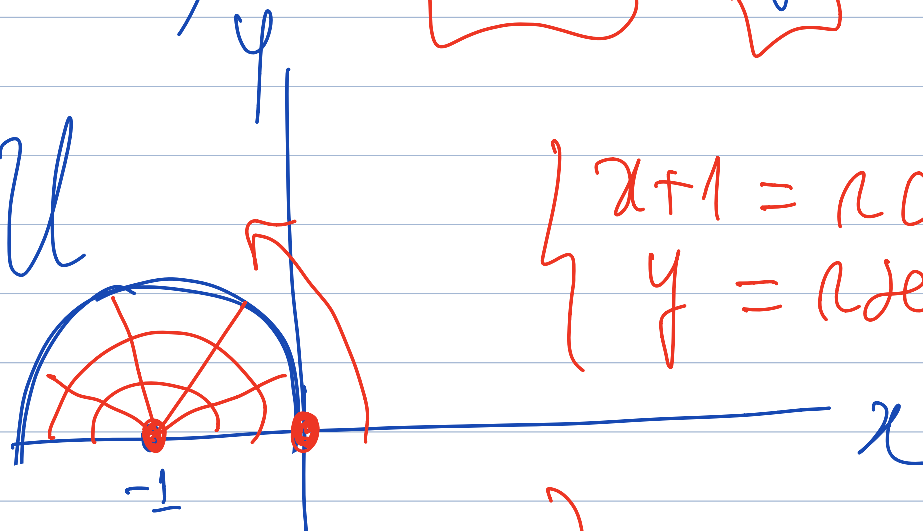
$(-2)r\,e^{-r^2} = \frac{d}{dr}e^{-r^2}$

$$= -\frac{1}{2}\int_0^{\pi/2}\left(e^{-r^2}\Big|_0^1\right)d\theta \quad \text{etc.}$$

2-c) $(x+1)^2+y^2<1,\ y>0$

$$\begin{cases} x+1 = r\cos\theta \\ y = r\operatorname{sen}\theta \end{cases}$$

U



$$\begin{cases} x = -1+r\cos\theta \\ y = r\operatorname{sen}\theta \end{cases}$$

r

$$\begin{cases} 0<\theta<\pi \\ 0<r<1 \end{cases}$$

$$\iint_U (x^2+y^2-1)\,dx\,dy = \int_0^\pi\int_0^1 \left[(r\cos\theta-1)^2+r^2\operatorname{sen}^2\theta-1\right] r \ dr\,d\theta$$

$$= \int_0^{\pi} \left(\int_0^1 \left(r^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta - \cancel{1} \right) r\,dr \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^1 \left(r^2 - 2r\cos\theta \right) r\,dr \right) d\theta$$

$$= \int_0^{\pi} \frac{1}{4}\,d\theta - 2\int_0^{\pi} \left(\int_0^1 r^2\cos\theta\,dr \right) d\theta$$

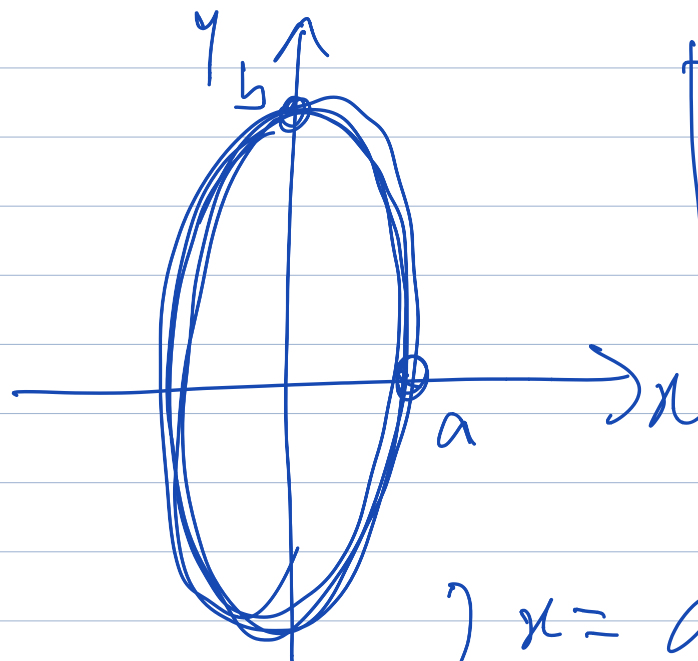
$$= \frac{\pi}{4} - 2\int_0^{\pi} \frac{1}{3}\cos\theta\,d\theta$$

etc

2 - e) $\boxed{\text{Elipse}}$

Coordenadas polares modificadas:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$$



$$\boxed{\begin{aligned} \frac{x}{a} &= r\cos\theta \\ \frac{y}{b} &= r\,\text{sen}\,\theta \end{aligned}}$$

$$\begin{cases} x = a\,r\cos\theta \\ y = b\,r\,\text{sen}\,\theta \end{cases}$$

$$(x,y) = g(r,\theta) = \left(ar\cos\theta,\ br\,\mathrm{sen}\,\theta\right)$$

$$\boxed{\det Dg(r,\theta) = abr}$$ (exercício)

$$u = \frac{x}{a}, \quad v = \frac{y}{b}$$

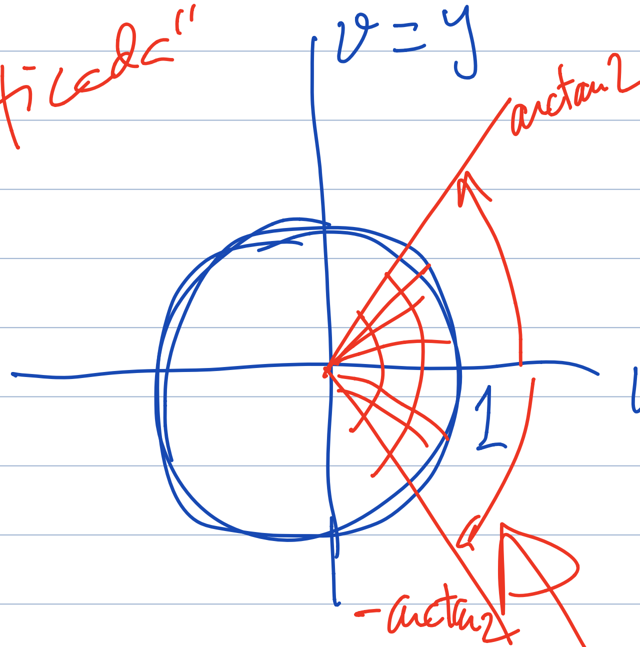
$$\boxed{u^2 + v^2 \leq 1}$$ círculo!!!

2-e) $\frac{x^2}{4} + y^2 < 1$; $\boxed{x > |y|}$

$$\left(\frac{x}{2}\right)^2 + y^2 < 1$$

$a = 2 \qquad b = 1$

"Modificada"



$u^2 + v^2 < 1$

$x = |y|$

$2u = |y|$

$2u = |v|$

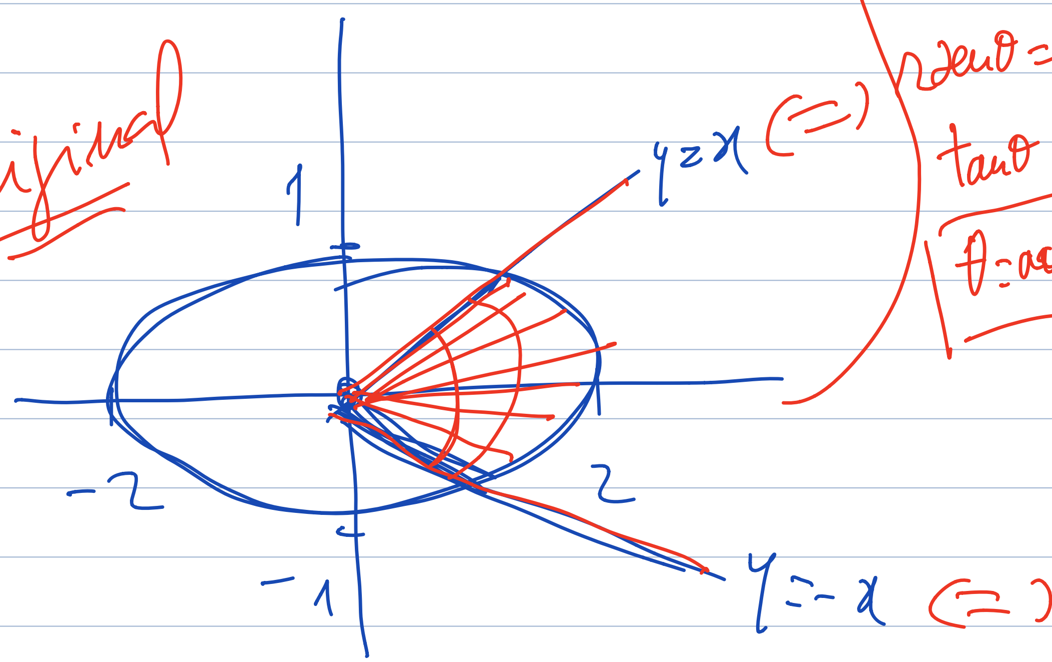
$$\begin{cases} x = 2r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$

$r\,\text{sen}\,\theta = 2r\cos\theta$

$\tan\theta = 2$

$\boxed{\theta = \arctan 2}$

original



$\theta = -\arctan 2$

$$\mathrm{Vol}_2(A) = \int_{-\arctan 2}^{\arctan 2} \left(\int_0^1 2r\,dr \right) d\theta$$

$$= 2\arctan(2) \quad //.$$

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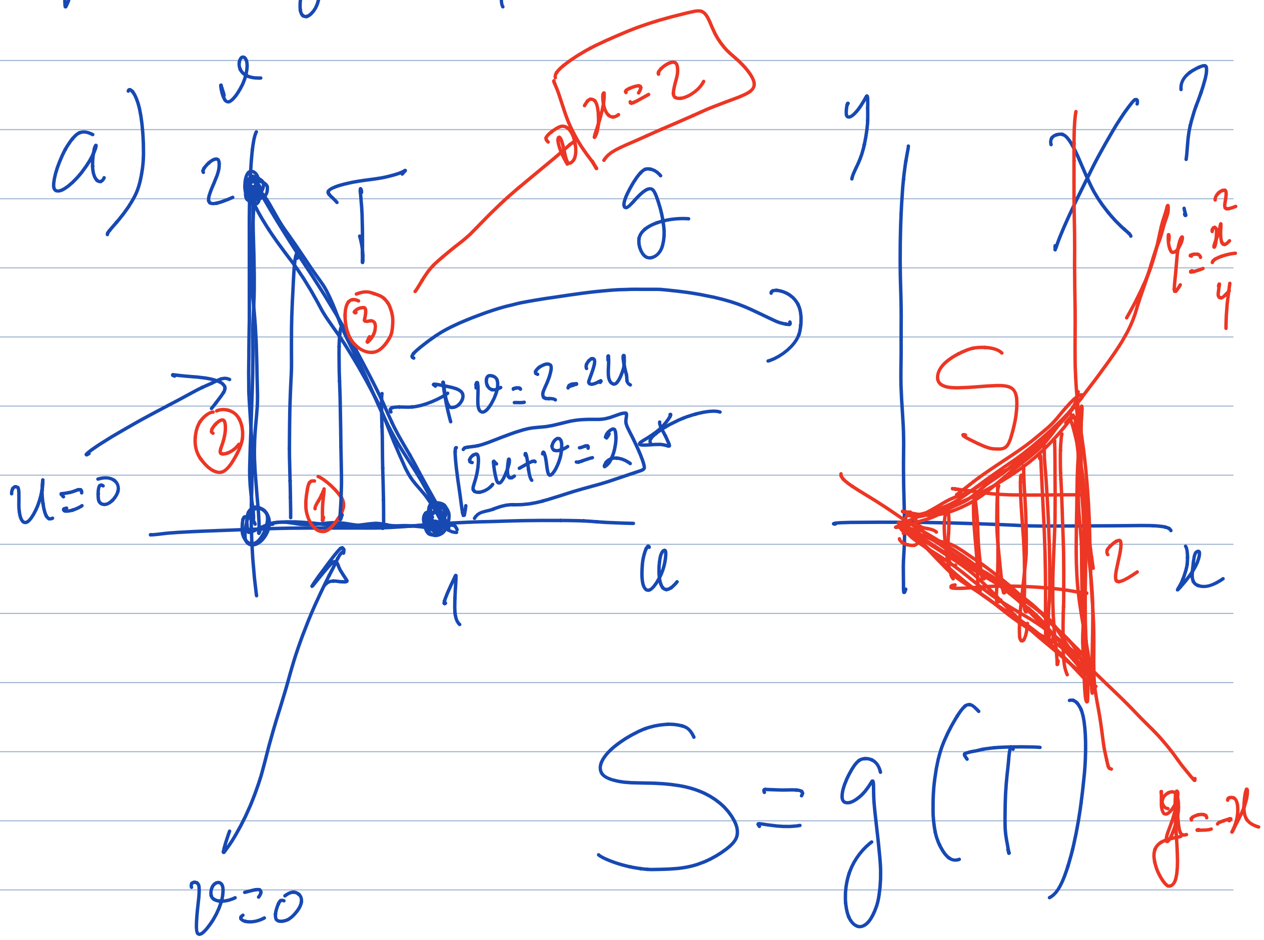
3 — $\begin{cases} x = \boxed{2u+v} \\ y = u^2 - v \end{cases}$

$x+y+1$
$= 2u+v+u^2-v+1$
$= u^2+2u+1$
$= (u+1)^2$

$$(x,y) = g(u,v) = (2u+v,\ u^2-v)$$

$$\det Dg(u,v) = \det\begin{bmatrix} 2 & 1 \\ 2u & -1 \end{bmatrix} = -2-2u$$

$$\left|\det Dg(u,v)\right| = \left|-2-2u\right| = 2+2u$$



$$S = g(T)$$

1 $\Downarrow$

$$\begin{cases} x = 2u \\ y = u^2 \end{cases} \qquad \begin{cases} u = \frac{x}{2} \\ \boxed{y = \frac{x^2}{4}} \end{cases}$$

2 $u = 0$

$\Downarrow$

$$\begin{cases} x = v \\ y = -v \end{cases}$$

$$\boxed{y = -x}$$

3-b) $$\iint_S \frac{1}{\sqrt{x+y+1}}\,dx\,dy \qquad f(x,y)$$

$$= \iint_T f\big(g(u,v)\big)\big(2+2u\big)\,du\,dv$$

$$= \int_0^1 \left(\int_0^{2-2u} \frac{1}{\cancel{\sqrt{(u+1)^2}}}\, 2\cancel{(1+u)}\,dv \right) du$$

$$= 2\int_0^1 \left(\int_0^{2-2u} 1\,dv \right) du = 2\int_0^1 (2-2u)\,du$$

etc....